Covariant Phase Space

Classical Field Theory Done Right

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Prolegomena to Any Future Physics

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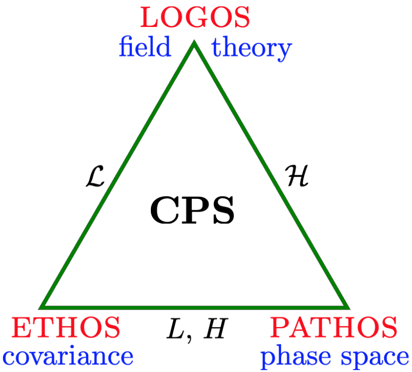

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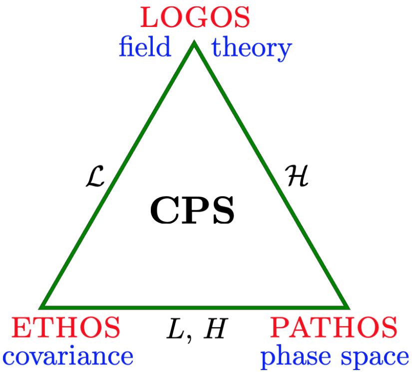

Theorem (my strongest conviction)

Both the Lagrangian (variational) and Hamiltonian (symplectic) formalisms work best when all three of these principles are unified.

Executive Summary

In pursuit of a covariant phase space for field theories, we will explore (a refined version of) the following hieroglyphs.

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The **variational principle** and the **symplectic potential**:

$$\delta\mathcal{L} = \mathcal{E}\,\delta\phi + \nabla_\mu\theta^\mu, \qquad \theta^\mu = \pi^\mu\,\delta\phi. \tag{0.1}$$

The **symplectic form** and **Hamilton's equations**:

$$\omega^\mu = \delta\theta^\mu = \delta\pi^\mu \wedge \delta\phi, \qquad \Omega = \int_\Sigma n_\mu\omega^\mu, \qquad \iota_{X_\xi}\Omega = \delta\mathcal{H}. \tag{0.2}$$

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Applications: CPS is unreasonably beautiful and unifies classical physics. It reproduces the ADM mass and reveals BH entropy as a Noether charge. It has the capacity to understand the phase space of GR, whose degrees of freedom live "on the boundary."

Outline

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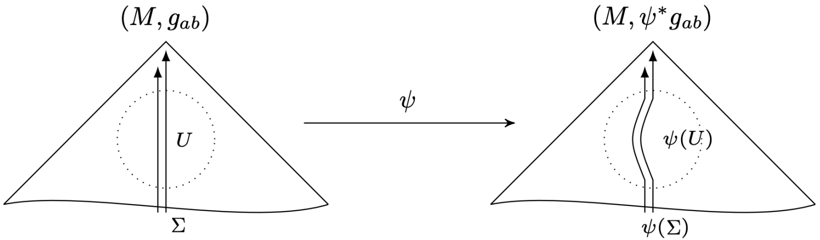
The Hole Argument

Suppose that we solve the initial value problem in GR for $g(x)$.

Perform a coordinate transformation $\psi\colon M \longrightarrow M$, sending $x \mapsto y = \psi(x)$, which leaves the initial value surface Σ fixed.



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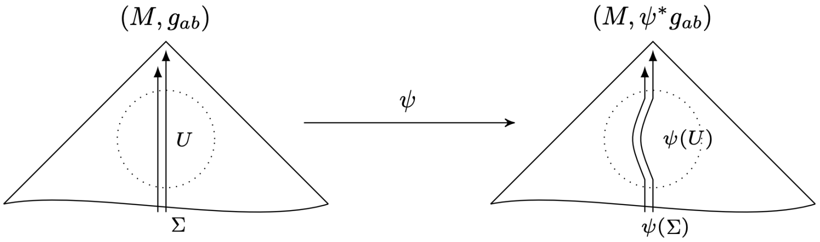
By coordinate invariance, $g(y) = (\psi^* g)(x) \neq g(x)$ must also solve the same initial value problem. Thus g is not determined uniquely!



The Hole Argument

The problem: GR seems to be indeterministic.

The resolution (physics): the solutions (M, g) and $(M, \psi^* g)$ are gauge-equivalent by the active diffeomorphism $\psi \in \mathrm{Diff}(M)$.

The resolution (math): the spacetimes (M, g) and $(\psi(M), \psi^* g)$ are isometric by ψ, by virtue of pulling back the metric.

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The moral: one must be careful when speaking of time, since the concept is generally meaningless. The initial value problem is not a covariant notion, and can be approached only in special cases.

Warning: this applies equally to *all* spacetimes, not just the weird ones. (And by the way, AdS is not even globally hyperbolic.)

The Problem of Time

The Hamiltonian in GR is zero. Three ways to see this:

1. **Physics:** the evolution of g is locally indistinguishable from a gauge transformation, which has vanishing Noether charge.
2. **Math:** $H = 0$ for *any* reparametrization-invariant system.
3. **Philosophy:** g is dynamical; it *creates* spacetime. There is no prior geometry, so g cannot evolve with a parameter it makes.

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We conclude that time in GR does not flow; it just *is*.

Meanwhile, unitarity in QM demands an absolute, rigid, external notion of time: $U = e^{-i\hat{H}t}$, and $|\psi(t)\rangle = U(t)\,|\psi_0\rangle$. As long as QM embraces an evolution parameter, it cannot be fully covariant.

The Wheeler-DeWitt Equation

Blindly quantizing yields the **Wheeler-DeWitt equation**, the Schrödinger equation for the quantum state of the universe:

$$\hat{H} \left|\Psi\right\rangle = 0. \tag{1.1}$$

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QM issues: The WDE has no classical limit (i.e. no $\hbar$). The state $|\Psi\rangle$ is "frozen" and cries out for a background-independent QM.

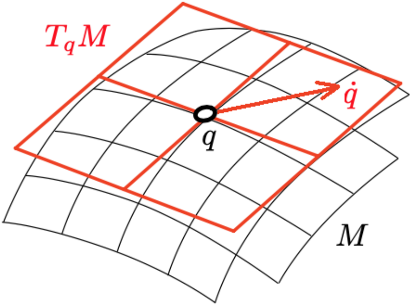
GR issues: H is the wrong thing to consider! We need a covariant object that generates the phase space flow of the metric.

We turn to the classical phase space of field theory and of gravity. Is there any more noble goal than to geometrize geometry?

Outline

Trajectories and Velocities

Let $M \subset \mathbb{R}^n$ be the world. The **trajectory** of a particle is a curve $q\colon \mathbb{R}_t \longrightarrow M$, and local coordinates $q^i \in \mathbb{R}^n$ describe its position.



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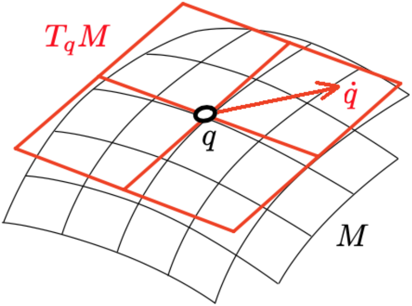
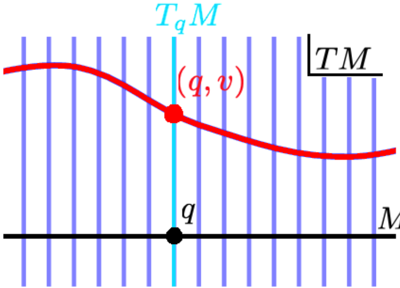
Its velocity is a vector (q, v) in the **tangent space** T_qM, which has a natural basis $\left\{\frac{\partial}{\partial q^i} = \partial_i\right\}$. Thus $v = v^i \frac{\partial}{\partial q^i} = v^i \partial_i$.





Lagrangian Mechanics I

"Why are q and $\dot{q}$ treated as independent?"

- The numbers v^i are coefficients needed to specify an arbitrary $v \in T_q M$, and can be chosen independently of q^i (duh).
- But when v is actually tangent to the trajectory, $v^i = \dot{q}^i(t)$.
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The initial data $(q_0, \dot{q}_0)$ uniquely determine $q(t)$. But they also determine $\dot{q}(t)$. So $q(t)$ and $\dot{q}(t)$ effect each other's dynamics.

Hence we are interested in the particle's combined trajectory $(q, \dot{q})\colon \mathbb{R}_t \longrightarrow TM$ traced out through the tangent bundle TM.

How does one determine this trajectory?

Lagrangian Mechanics II

The **Lagrangian** of a mechanical system is a function $L\colon TM \longrightarrow \mathbb{R}$, and the **action** functional is its integral over $\mathbb{R}_t$:

$$S[q(t), \dot{q}(t)] = \int_{\mathbb{R}} \mathrm{d}t\, L(q, \dot{q}). \tag{2.1}$$

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By requiring that $\delta q = 0$ at infinity, the **Euler-Lagrange (EL) equations** follow. In local coordinates on TM, these are

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0. \tag{2.2}$$

E.g. For a free particle on (M, g), the Lagrangian is the metric, $L(x, \dot{x}) = \frac{1}{2} g(\dot{x}, \dot{x}) = g_{ij}\dot{x}^i\dot{x}^j$. The EOM is the geodesic equation.

Upgrading to Differential Forms

Everything in sight is now a **differential form**—an antisymmetric tensor—on the parameter space $\mathbb{R}_t$ (time, basis $\mathrm{d}t$, "horizontal") as well as on the target space M (space, basis δq^i, "vertical").

E.g. $L = \mathcal{L}\,\mathrm{d}t$ is a 1-form on $\mathbb{R}_t$ and a 0-form on M. The action $S = \int_{\mathbb{R}} L$ is a scalar. The variation δL is then a $(1,1)$-form.

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Our main tools: $\mathrm{d}^2 = \delta^2 = 0$ ("differential forms are fermions") and **Stokes's theorem**, $\int_M \mathrm{d}\omega = \int_{\partial M} \omega$ (duality of d and ∂).

E.g. when the EOM hold, the variation of L must either vanish or be a total time derivative with vanishing integral over $\mathbb{R}_t$:

$$\delta L = (\delta\mathcal{L})\mathrm{d}t = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)\mathrm{d}t = \mathrm{d}\sigma, \qquad \int_{\mathbb{R}} \mathrm{d}\sigma = \int_{\partial\mathbb{R}} \sigma = 0. \qquad (2.3)$$

The Fundamental Calculation

Consider a single particle moving in one dimension. We vary L:

$$\delta L = (\delta\mathcal{L})\mathrm{d}t = \left[\frac{\partial\mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)\right]\delta q\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\,\delta q\right)\mathrm{d}t \equiv$$

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At the level of the action, this calculation reads

$$\delta S = \int_{\mathbb{R}} \delta L = \int_{\mathbb{R}} \mathcal{E}\,\delta q\,\mathrm{d}t + \int_{\partial\mathbb{R}} p\,\delta q = 0. \tag{2.5}$$

Main Results of Lagrangian Mechanics

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2. If δq is an **on-shell symmetry**, i.e. $\delta S = 0$ when $\mathcal{E} = 0$,

$$\delta L = \cancel{\mathcal{E}\,\delta q\,\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\big(p\,\delta q\big)\mathrm{d}t = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\mathrm{d}t \implies \frac{\mathrm{d}}{\mathrm{d}t}\Big(p\,\delta q - \sigma\Big) = 0.$$

We call $p\,\delta q - \sigma$ the **Noether current** of the symmetry δq.

Example: The Harmonic Oscillator

The configuration space is $\mathbb{R}_x$, and the tangent bundle is $\mathbb{R}^2_{(x,\dot{x})}$.

The Lagrangian is $L(x,\dot{x}) = \mathcal{L}\,\mathrm{d}t = \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right)\mathrm{d}t$, so

$$\delta L = \underbrace{\left[-m\omega^2 x - m\ddot{x}\right]}_{\mathcal{E}}\delta x\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(m\dot{x}\,\delta x\right)}_{\theta}\mathrm{d}t. \qquad (2.7)$$

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The equations of motion are

$$\mathcal{E} = -m\omega^2 x - m\ddot{x} = 0 \iff \ddot{x} = -\omega^2 x, \qquad (2.8)$$

and the symplectic potential is $\theta = p\,\delta x = m\dot{x}\,\delta x$.

Example: The Harmonic Oscillator

A Noether current is obtained for each symmetry δx of the action.

E.g. $\delta x = \dot{x} \implies \delta\dot{x} = \ddot{x}$ generates time translation. We have:

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Thus $\delta L = \mathrm{d}\mathcal{L}$ and $\theta = m\dot{x}\,\delta x = m\dot{x}^2 = p\,\delta x$, so $\mathcal{H}$ is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(p\,\delta x - \mathcal{L}\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2\right) = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0. \quad (2.10)$$

We have "discovered" the Legendre transformation of $\sigma = \mathcal{L}$ via θ.

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2. **Conservation of energy.** Consider a Hamiltonian vector field. Physically motivated, but not (immediately) symplectic.
3. **Construct phase space.** Use θ to build the symplectic form. Possibly illuminating, but is a long story and takes effort.
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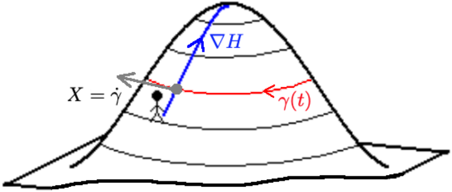
The uncomfortable truth: it works because it works.

Hamilton's Equations for Dummies

Big idea: The phase space trajectory $\gamma(t) = \big(q(t), p(t)\big)$ is an integral curve of the **Hamiltonian vector field** $X = (\dot{q}, \dot{p})$.



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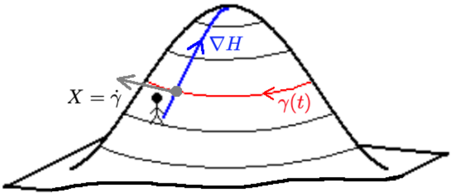
$$(\dot{q}, \dot{p}) = X \perp \nabla H = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right). \tag{2.11}$$



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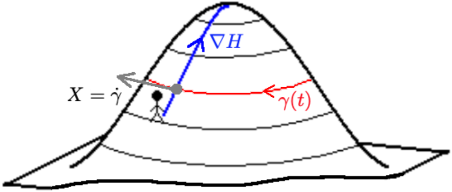
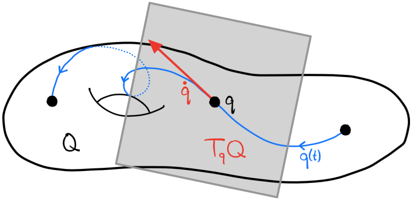
Therefore the components of X must be $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$.



What is Phase Space?

Phase space is the set of all initial conditions for the EOM, or equivalently all solutions.



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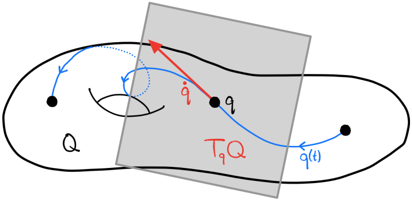

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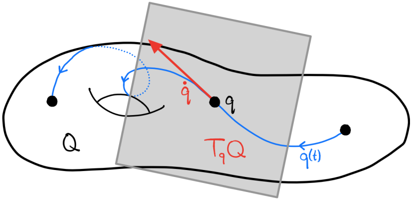
The action of p on $v \in TM$ is

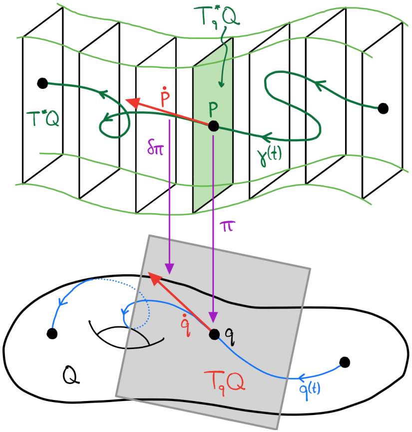
$$p(v) = (p\,\delta q)\left(\dot{q}\frac{\partial}{\partial q}\right) = p\dot{q}.$$

If $v = \frac{\mathrm{d}}{\mathrm{d}t}q(t)$, this is θ again!



What is Phase Space?

The **Hamiltonian vector field** $X \in T\mathcal{M}$ is the "velocity" tangent to $\gamma(t)$, and Hamiltonian mechanics is the flow by X.



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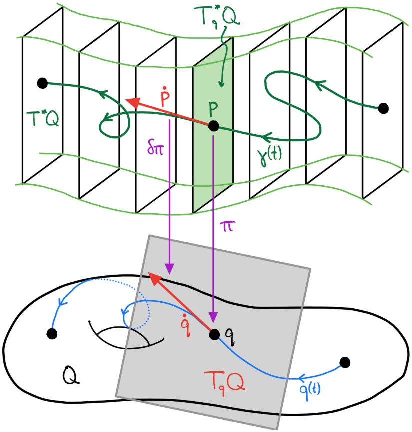

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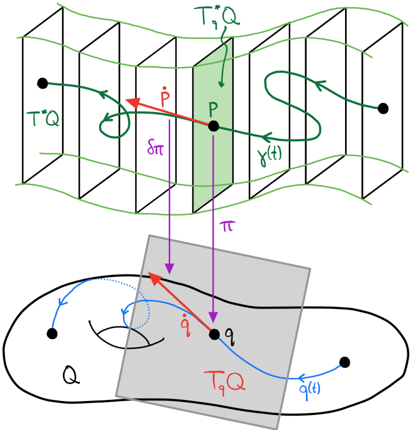
Since θ is also $p\,\delta q$, it is the bridge between the Lagrangian and Hamiltonian viewpoints.



The Symplectic Form

We are now in a position to use $\theta = p\,\delta q$ to relate X to $H = \mathcal{H}\,\mathrm{d}t$. The key insight is that $X \perp \nabla\mathcal{H} \sim \delta\mathcal{H}$; to make this precise, we seek an antisymmetric machine that raises and lowers indices.

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The **symplectic form** is $\omega = \delta\theta = \delta p \wedge \delta q$. It is closed, $\delta\omega = 0$, and nondegenerate, i.e. $\iota_X\omega \equiv \omega(X, -)$ is a 1-form unique to X.

Now $\omega(X, -)$ "lowers the index" of X, sends $(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}) \mapsto (\delta q, \delta p)$, and rotates its entries by $\frac{\pi}{2}$. (Un)surprisingly, the result is $-\delta\mathcal{H}$:

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$$X = (\dot{q}, \dot{p}) \implies \iota_X\omega = (-\dot{p}, \dot{q}) = -\delta\mathcal{H} = \left(-\frac{\partial\mathcal{H}}{\partial q}, -\frac{\partial\mathcal{H}}{\partial p}\right). \quad (2.12)$$

Thus **Hamilton's equations** are expressed by $\iota_X\omega + \delta\mathcal{H} = 0$.

Example: The Harmonic Oscillator

The configuration space is $\mathbb{R}_t$, and the phase space is $\mathbb{R}^2_{(p,x)}$.

The Hamiltonian is $H(p,x) = \left(\frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2\right)\mathrm{d}t = \mathcal{H}\,\mathrm{d}t$.

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$$\delta\mathcal{H} = \frac{\partial\mathcal{H}}{\partial x}\delta x + \frac{\partial\mathcal{H}}{\partial p}\delta p = m\omega^2 x\,\delta x + \frac{p}{m}\delta p,$$

$$X = (\dot{x},\dot{p}) = \dot{x}\frac{\partial}{\partial x} + \dot{p}\frac{\partial}{\partial p} \implies \iota_X\omega = -\dot{p}\,\delta x + \dot{x}\,\delta p. \qquad (2.13)$$

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Therefore $\iota_X\omega = -\dot{p}\,\delta x + \dot{x}\,\delta p \overset{!}{=} -m\omega^2 x\,\delta x - \frac{p}{m}\delta p = -\delta\mathcal{H}$,

and matching differentials yields $\dot{x} = \frac{p}{m}$ and $\dot{p} = -m\omega^2 x$. Nice!

Outline

Summary So Far

Lagrangian mechanics:

- $L = \mathcal{L}\,\mathrm{d}t$ lives on TM and determines the path $\big(q(t), \dot{q}(t)\big)$.
- The variational principle yields the EOM and Noether charges.
- The all-important symplectic potential $\theta = p\,\delta q \sim \delta S$ typically vanishes on $\partial\mathbb{R}$, but can be nonzero in the bulk.
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Hamiltonian mechanics:

- The Hamiltonian vector field X generates phase space flow and determines the path $\gamma(t) = \big(q(t), p(t)\big)$ through $\mathcal{H} = E$.
- We reimagine θ as the canonical 1-form on T^*M.
- The symplectic form encodes the structure of Hamilton's equations, and converts between X and H.

Two Schools of Thought

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- The field has one degree of freedom at every $\mathbf{x} \in M$, and the notion of particle positions evaporates: $M^M \longrightarrow M \times \mathbb{R}$.
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Fields are sigma models. *Space and time are unified.*

- A **field** $\phi\colon M \longrightarrow F$ maps spacetime points $x \in M$ to field values $\phi(x) \in F$, and has $\dim F$ degrees of freedom.
- This generalizes the time parameter and the configuration space of particle mechanics to arbitrary manifolds.

First Attempt: De Donder–Weyl Theory

Lagrangian field theory is already covariant: $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$.

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$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}, \qquad \mathcal{H} = \pi^\mu \partial_\mu \phi - \mathcal{L}. \tag{3.1}$$

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N.B. This $\mathcal{H}$ *is* covariant, but *does not* generate time translations; meanwhile, the "textbook" $\mathcal{H}$ *cannot* be covariant! Also, the DW theory has too many momenta. The CPS formalism soaks up the index in π^μ by choosing a Cauchy surface Σ and considering $\pi^\mu n_\mu$.

Too Many Bundles: Fields and their Jets

We generalize the tangent bundle, spanned by vectors $\dot{q}$, to a space spanned by *all* of field derivatives $\partial_\mu \phi$. This is the **jet bundle** $J^1 F$. We also want to consider both spacetime differentials $\mathrm{d}x$ and field variations $\delta\phi$, so we define the **field bundle** $E \longrightarrow M$ with fiber F.

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E.g. The real scalar field: $M = \mathbb{R}^{3,1}_{x^\mu}$, $F = \mathbb{R}_\phi$, and $E = \mathbb{R}^{3,1}_{x^\mu} \times \mathbb{R}_\phi$.
Then $(J^1 E)^* = \mathbb{R}^{3,1}_{x^\mu} \times \mathbb{R}_\phi \times \mathbb{R}^{3,1}_{\pi^\mu}$. Everything is finite-dimensional!

How Classical Physics Should Be Done

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The **multisymplectic potential** and **multisymplectic form** are

$$\theta = (\pi^\mu\,\delta\phi)n_\mu\,\varepsilon_{\partial M} \in \Omega^{(n-1,1)}, \quad \omega = \delta\theta \in \Omega^{(n-1,2)}. \tag{3.4}$$

The dream: do geometrical quantization to all of this!

Outline

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1. Begin with the **kinematic configuration space** $\mathcal{C}$ and its **dynamical shell** $\widetilde{\mathcal{P}}$, also called the pre-phase space.
2. Vary the action, taking care of boundary conditions, to obtain a pre-symplectic potential $\widetilde{\theta}$ and pre-symplectic form $\widetilde{\Omega} = \delta\widetilde{\theta}$.
3. Quotient out $\widetilde{\mathcal{P}}$ and $\widetilde{\Omega}$ by **gauge symmetries**, which are zero modes of $\widetilde{\Omega}$. This gives us the covariant phase space $(\mathcal{P}, \Omega)$.
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Once all of this is done, we will proceed to apply it to gravity!

Volume Forms and Boundaries

In moving from particles to fields, we put **volume form** on M:

$$\mathrm{d}t \longrightarrow \varepsilon_M = \sqrt{-g}\,\mathrm{d}^n x = \sqrt{-g}\,\mathrm{d}x_{\mu_1} \wedge \cdots \wedge \mathrm{d}x_{\mu_n}. \tag{4.1}$$

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E.g. On the half-Minkowski space $\mathbb{R}^{3,1}_{x \leq 0}$, we have

$$\begin{aligned} \varepsilon_M &= \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \\ n^\mu &= \partial^x = (0,1,0,0), \\ \varepsilon_{\partial M} &= n^\mu\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu\,\mathrm{d}x_\rho\,\mathrm{d}x_\sigma = -\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}z. \end{aligned} \tag{4.2}$$

Variation of the Lagrangian

We proceed as before: given $L = \mathcal{L}(\phi, \partial_\mu \phi)\, \varepsilon_M$, we have

$$\delta L = \left[\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi)}\right)\right] \delta\phi\, \varepsilon_M + \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi)} \delta\phi\right) \varepsilon_M =$$

$$= \mathcal{E}\, \delta\phi\, \varepsilon_M + \nabla_\mu(\pi^\mu\, \delta\phi)\varepsilon_M \stackrel{!}{=} (\nabla_\mu \sigma^\mu)\varepsilon_M. \tag{4.3}$$

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We get a vector's worth of symplectic potentials $\theta^\mu = \pi^\mu\, \delta\phi$ and variations σ^μ. The "product rule" gives us their boundary values:

$$(\nabla_\mu \sigma^\mu) \varepsilon_M = \mathrm{d}\sigma, \qquad \sigma\Big|_{\partial M} = (n_\mu \sigma^\mu) \varepsilon_{\partial M},$$
$$(\nabla_\mu \theta^\mu) \varepsilon_M = \mathrm{d}\theta, \qquad \theta\Big|_{\partial M} = (n_\mu \theta^\mu) \varepsilon_{\partial M}. \tag{4.4}$$

As advertised, $\pi^\mu n_\mu$ is the "correct" momentum conjugate to ϕ.

Variation of the Action

At the level of the action, these divergences become surface terms:

$$\delta S = \int_M \mathcal{E}\,\delta\phi\,\varepsilon_M + \int_M (\nabla_\mu \theta^\mu)\varepsilon_M = E + \int_M \mathrm{d}\theta = \\ = E + \int_{\partial M} \theta = E + \int_{\partial M} (n_\mu \pi^\mu\,\delta\phi)\varepsilon_{\partial M}. \tag{4.5}$$

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If $\delta\phi$ vanishes on ∂M, then $\delta S = 0$ implies $\mathcal{E} = 0$ as usual.
And if $\delta\phi$ is an on-shell symmetry, we get Noether's theorem:

$$\delta L = (\nabla_\mu \theta^\mu)\varepsilon_M = (\nabla_\mu \sigma^\mu)\varepsilon_M \implies \nabla_\mu(\theta^\mu - \sigma^\mu) = 0. \quad (4.6)$$

Thus the **Noether current** $j^\mu = \pi^\mu\,\delta\phi - \sigma^\mu$ is conserved.

Symplectic Circus

The "full" symplectic potential θ and form ω are defined by contracting θ^μ and $\omega^\mu = \delta\theta^\mu$ into ε_M:

$$\begin{aligned} \theta^\mu = \pi^\mu\,\delta\phi \implies \theta = \iota_{\theta^\mu}\varepsilon_M \longrightarrow (n_\mu \pi^\mu\,\delta\phi)\varepsilon_{\partial M}, \\ \omega^\mu = \delta\pi^\mu \wedge \delta\phi \implies \omega = \iota_{\omega^\mu}\varepsilon_M \longrightarrow (n_\mu\,\delta\pi^\mu \wedge \delta\phi)\varepsilon_{\partial M}. \end{aligned} \tag{4.7}$$

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We restrict to globally hyperbolic M, choose a Cauchy surface Σ, call ω the **symplectic density**, and define the **symplectic form**

$$\Omega = \int_\Sigma \omega = \int_\Sigma (\hat{n}_\mu \omega^\mu)\varepsilon_\Sigma. \qquad (\Omega_\Sigma = \Omega_{\Sigma'}) \tag{4.8}$$

where $\hat{n}$ is the (past-pointing) normal to Σ. This is still covariant!

The Lagrangian and its Variation

The real, free scalar field on Minkowski spacetime $M = \mathbb{R}^{3,1}$ has phase space $(J^1E)^* = \mathbb{R}^{3,1}_{x^\mu} \times \mathbb{R}_\phi \times \mathbb{R}^{3,1}_{\pi^\mu}$ and Lagrangian

$$L = \mathcal{L}\,\mathrm{d}^4x = -\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2}m^2\phi^2\right]\mathrm{d}^4x. \qquad (4.9)$$

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The conjugate momenta are $\pi^\mu = \partial^\mu\phi$. We vary the Lagrangian to obtain the equations of motion and the symplectic data:

$$\delta L = \underbrace{\left[\left(\partial_\mu\partial^\mu - m^2\right)\phi\right]}_{\mathcal{E}}\,\delta\phi\,\mathrm{d}^4x + \partial_\mu\underbrace{\left(\partial^\mu\phi\,\delta\phi\right)}_{\theta^\mu}\,\mathrm{d}^4x. \tag{4.10}$$

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The equations of motion are $\left(\partial_\mu\partial^\mu - m^2\right)\phi = 0$, and we have

$$\theta^\mu = \partial^\mu\phi\,\delta\phi = \pi^\mu\,\delta\phi \implies \omega^\mu = \delta\theta^\mu = \delta\pi^\mu \wedge \delta\phi. \tag{4.11}$$

The Noether Current

Consider the generator of spacetime translations:

$$\delta_\nu \phi = \partial_\nu \phi = \pi_\nu \implies \delta_\nu (\partial_\mu \phi) = \partial_\nu \partial_\mu \phi = \partial_\nu \pi_\mu . \tag{4.12}$$

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$$\begin{aligned} \delta_\nu L = & \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] \varepsilon_M = \\ & = - \Big[\big(m^2 \phi \big) (\partial_\nu \phi) + (\partial^\mu \phi)(\partial_\nu \partial_\mu \phi) \Big] \varepsilon_M = \\ & = - \partial_\mu \left[\frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2 \right] \varepsilon_M = (\partial_\nu \mathcal{L}) \varepsilon_M . \end{aligned} \tag{4.13}$$

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The conserved current is evidently the **stress tensor**:

$$j^\mu_\nu = \theta^\mu_\nu - \sigma^\mu_\nu = \partial^\mu \phi \, \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} = T^\mu_\nu . \tag{4.14}$$

Removing Gauge Symmetries

If two nearby field configurations ϕ and $\phi + \delta\phi$ represent the same physical state, then the vector Z "$=$" $\delta\phi$ is a degenerate direction in *pre*-phase space $\widetilde{\mathcal{P}}$, and the *pre*-symplectic form $\widetilde{\Omega}$ is degenerate.

(More precisely, $\delta\phi = \mathscr{L}_Z \phi$, where $\mathscr{L}$ is the Lie derivative Lie derivative. **Lie derivative**.)

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We formally glue all equivalent field configurations along all Z to obtain the phase space $\mathcal{P} = \widetilde{\mathcal{P}}/G$. We also obtain the symplectic form $\Omega = \widetilde{\Omega}/G$ by gluing vector fields that differ by a zero mode.

Covariance and Symmetry

In practice, one specifies $\mathcal{H}$ and uses Ω to compute X, which effects phase space evolution. Since we have Ω, we are "done".

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- If $S[q(t), \dot{q}(t)]$ is invariant under $t \longrightarrow t + \varepsilon$, the induced transformation $q(t) \longrightarrow q(t+\varepsilon)$ on phase space is generated by $\delta q = q(t+\varepsilon) - q(t) = \varepsilon \dot{q}$ and has Noether charge $\mathcal{H}$.
- If $S[\phi(x), \partial_\mu \phi]$ is invariant under $x \longrightarrow x + \varepsilon$, the induced phase-space symmetry $\delta_\mu \phi = \varepsilon \partial_\mu \phi$ yields the eigenvalues $\mathcal{H}^\mu$ of the stress tensor $T^{\mu\nu}$ as Noether charges.
- More generally, any transformation $x \longrightarrow x'$ that generates a symmetry $\delta_\xi \phi$ has a corresponding Noether charge.

Hamiltonian Vector Fields

But in GR, coordinate transformations on M are gauged and yield vanishing Noether charges, except when δg arises from an isometry.

Thus we ask: how do we obtain a Hamiltonian vector field and its Noether charge for symmetries of $\mathcal{P}$ generated by isometries of M?

Hamiltonian Vector Fields

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Thus we ask: how do we obtain a Hamiltonian vector field and its Noether charge for symmetries of $\mathcal{P}$ generated by isometries of M?

The answer: if ξ is an isometry, the Hamiltonian vector field is

$$X_\xi = \left(\int_M \mathscr{L}_\xi \phi^a\right) \frac{\delta}{\delta \phi^a} = \left(\int_M \delta_\xi \phi^a\right) \frac{\delta}{\delta \phi^a} \in T\mathcal{P}. \qquad (4.15)$$

This vector field implements the flow of ξ *only* on the dynamical fields ϕ in $\mathcal{P}$, and does *not* flow the rest of the gunk in spacetime.

Diffeomorphism Charges and Hamilton's Equations

The Noether current for X_ξ is essentially just $\theta - \sigma$. More precisely, $J_\xi = \iota_{X_\xi}\theta - \iota_\xi$. (This is a souped-up version of $H - p\dot{q} - L$.)

Finally, we seek the "Hamiltonian" $\mathcal{H}_\xi$ for which $\iota_{X_\xi}\Omega = -\delta\mathcal{H}_\xi$.

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To find it, we use the explicit forms of X_ξ and Ω to compute $\iota_{X_\xi}\Omega$. If the result is $\delta(☺)$, then "☺" is our H_ξ. Indeed,

$$H_\xi = \int_\Sigma J_\xi + \int_{\partial\Sigma} \left(\iota_\xi \delta\ell - \iota_{X_\xi} C\right), \tag{4.16}$$

where ℓ is (!) the Lagrangian on ∂M.

Outline

The Action and its Variation

Let (M, g) have boundary $(\partial M, \gamma)$. The full gravity action consists of the **Einstein-Hilbert** and **Gibbons-Hawking-York** terms:

$$
\begin{aligned}
S = S_{\mathrm{EH}} + S_{\mathrm{GHY}} &= \int_M L + \int_{\partial M} \ell = \\
&= \frac{1}{16\pi G} \int_M R\, \varepsilon_M + \frac{1}{8\pi G} \int_{\partial M} K\, \varepsilon_{\partial M} .
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The variation of L leads to the Einstein field equations:

$$\delta L = \mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + \mathrm{d}\Theta, \quad \mathcal{E}^{\mu\nu} = \frac{1}{16\pi G} \left(-R^{\mu\nu} + \frac{1}{2} R g^{\mu\nu} \right) \varepsilon_M. \qquad (5.2)$$

Meanwhile, $\delta \ell = \frac{1}{16\pi G} (\text{stuff}) \varepsilon_{\partial M}$ contributes to Θ on ∂M.

The Symplectic Potential and Form

After a short calculation, we obtain

$$\begin{aligned}(\Theta+\delta\ell)\Big|_{\partial M} &= -\frac{1}{16\pi G}(K^{\mu\nu}-K\gamma^{\mu\nu})\delta g_{\mu\nu}\,\varepsilon_{\partial M}+\mathrm{d}C= \\ &= \frac{1}{2}T^{\mu\nu}_{\mathrm{BY}}\,\delta g_{\mu\nu}\,\varepsilon_{\partial M}+\mathrm{d}C, \\ C &= -\frac{\gamma^{\mu\nu}n^{\alpha}\delta g_{\nu\alpha}}{16\pi G}\cdot\varepsilon_{\partial M}\neq 0 \end{aligned} \tag{5.3}$$

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The boundary-corrected symplectic potential in GR consists of the **Brown-York stress tensor** and *another* total divergence.

Taking $\delta g_{\mu\nu}\Big|_{\partial M} = 0$, i.e. fixing the metric on ∂M, does *not* set the boundary term in δS to zero! See [Harlow-Wu 2019] for an explanation of why such terms should generally be present.

Commentary

Once we allow for a nonzero flux from $\mathrm{d}C$ in the on-shell variation

$$\delta S = \int_{\partial M} (\Theta + \delta\ell) = \int_{\partial M} \left(\frac{1}{2} T^{\mu\nu}_{\mathrm{BY}} \delta g_{\mu\nu} + \mathrm{d}C \right) \varepsilon_{\partial M}, \qquad (5.4)$$

the Dirichlet boundary conditions $\delta\gamma = 0$ render the variational problem in GR well-posed in a covariant way. Viewing γ as a fixed source reminds one of the **extrapolate dictionary** in AdS/CFT.

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By Stokes's theorem, the boundary-of-a-boundary term C lives on the codimension-2 **corners** of the spacetime. Holography, anyone?

There are also lines of research investigating "edge modes" and "corner potentials" in gravity that are somewhat related.

Diffeomorphism Charges

After some inspiration by Wald and a straightforward calculation of Harlow-Wu, one finds the diffeomorphism charges of GR:

$$J_\xi = \mathrm{d}Q_\xi \implies H_\xi = -\int_{\partial\Sigma} \tau^\mu \xi^\nu T^{\mathrm{BY}}_{\mu\nu} \varepsilon_{\partial\Sigma}. \tag{5.5}$$

This is the expression for the generators of boundary isometries with Killing field ξ^μ, and is once again a corner term.

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Commentary: CPS is powerful and recovers hard results in GR (ADM, BY, even S_{BH}) with relative ease. It smells a lot like holography ("AdS/CFT is just spicy Stokes's theorem"), and is way too beautiful *not* to be immediately adopted by everyone.

Summary and Conclusions

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